Schmitt trigger: A solvable model of stochastic resonance

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An ideal Schmitt trigger (ST) is the simplest two-state system available for realization of stochastic resonance (SR). An exact solution is found for the ST driven by exponentially correlated Gaussian noise plus a weak periodic signal. The signal-to-noise ratio is shown to pass through a maximum at a specific value of the noise intensity. A qualitative classification is given for a class of noise models which either exhibit SR or do not.

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I. INTRODUCTION

Many physical phenomena can be interpreted in terms of transitions of a system between two or more metastable states under the effect of random noise. Benzi and co-workers [1] and Nicolis [2] used this concept to explain the onset of the Earth's ice ages and the intervening periods of relative warmth as a cooperative effect between the weak, periodic variations of the eccentricity of the Earth's orbit and random fluctuations in the solar insolation. They discovered that a weak periodic signal can be dramatically increased in strength relative to the noise background when the noise intensity is "tuned" to a particular value.

The effect was first demonstrated in a laboratory experiment with an electronic device—a Schmitt trigger (ST) by Fauve and Heslot [3]—and, more recently, with a bistable ring laser [4]. These experiments stimulated a great deal of theoretical activity [5–12], a further experiment [13], and analog simulations [8,9,12,14,15]. Further information can be found in some recent workshop proceedings [16,17] and a review [18]. The common feature of all these systems, theoretical models, as well as experimental realizations, is their strong nonlinearity, which renders them solvable only by various approximate procedures supported by numerical simulations. We point out that the *ideal* ST is different from many other experimental realizations. First, it is an ideal two-state device: its output can be either of only two values. Second, assuming that the switching time is fast enough that it can be neglected compared to all other time scales, which is often the case in practical applications, the ST is not a dynamical system; rather it is simply a bistable threshold device.

The purpose of this paper is to give an analytical solution for the Schmitt trigger pumped by a weak signal and a colored noise of arbitrary intensity. The results of this calculation provide us with analytical expressions for the statistical characteristics of the ST and for the weak-signal amplification coefficient. We hope that the rigorous solution of this simple system will be instructive for the investigation of more realistic systems.

II. GENERAL EQUATIONS

The ST is an electronic circuit with the output fixed to one of two voltages and a hysteresis—there is a range of the input for which the circuit is bistable. A circuit diagram of the Schmitt trigger is presented in Fig. 1(a), its input-output characteristic is shown in Fig. 1(b). The trigger resides in state 1 as long as the input voltage V is less than V_0. At $V=V_0$ the trigger switches instantaneously into state 2 and resides in it as long as $V>-V_0$. The input voltage can be divided into a signal and noise components. For our purposes it is sufficient to consider a weak sinusoidal signal and a Gaussian noise,

$$V(t)=v(t)+\epsilon V_0\cos\Omega_s t \ . \tag{1}$$

In order to restrict variations of $v(t)$ in a given interval of time Δt, we assume $v(t)$ is a colored noise with a correlator

$$\langle v(t)v(t')\rangle=\frac{\sigma^2}{2\tau_c}\exp\left[-\frac{|t-t'|}{\tau_c}\right] , \tag{2}$$

where σ gives the integral dispersion of the noise, τ_c is the correlation time. The random function $v(t)$ obeys the equation

$$\frac{dv}{dt}=-\frac{v}{\tau_c}+\frac{\sigma}{\tau_c}\xi(t) , \tag{3}$$

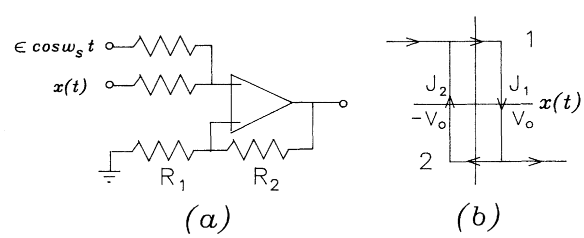

FIG. 1. (a) A circuit diagram of the Schmitt trigger. The threshold V_0 is determined by the ratio of the resistances R_1/R_2. (b) The input-output characteristic of the trigger. The noise coordinate is the x axis, and $J_{1,2}$ are the probability fluxes.

where $\xi(t)$ is a δ-correlated Gaussian noise,

$$\langle \xi(t)\xi(t') \rangle = \delta(t-t') . \tag{4}$$

To simplify calculations, we rescale v by V_0,

$$x \equiv v/V_0 , \tag{5}$$

and substitute

$$t \to t/\tau_c . \tag{6}$$

Then the input of the trigger becomes

$$X(t) = x(t) + \epsilon \cos\omega_s t , \tag{7}$$

where $\omega_s \equiv \Omega_s \tau_c$,

$$\frac{dx}{dt} = -x + \frac{\sigma}{V_0 \tau_c^{1/2}} \xi(t) . \tag{8}$$

The trigger switches instantaneously from the state 1 into the state 2 as soon as $X(t)$ reaches the value $X=1$ and backwards at $X(t)=-1$. The Langevin equation (8) can be substituted by an equivalent Fokker-Planck equation,

$$\frac{\partial P}{\partial t} = \frac{\partial}{\partial x}\left[D\frac{\partial P}{\partial x} + xP \right] , \tag{9}$$

where the effective diffusion coefficient is introduced,

$$D \equiv \frac{\sigma^2}{2V_0^2 \tau_c} . \tag{10}$$

For distribution functions $P_{1,2}(t,x)$, which completely specify the state of an ensemble of the triggers, one can write the system

$$\frac{\partial P_1}{\partial t} = \frac{\partial}{\partial x}\left[D\frac{\partial P_1}{\partial x} + xP_1 \right] + D\delta(x+\epsilon\cos\omega_s t + 1)\frac{\partial P_2}{\partial x} , \tag{11}$$

$$\frac{\partial P_2}{\partial t} = \frac{\partial}{\partial x}\left[D\frac{\partial P_2}{\partial x} + xP_2 \right] - D\delta(x+\epsilon\cos\omega_s t - 1)\frac{\partial P_1}{\partial x} , \tag{12}$$

where the last terms describe the switching events between the two states. This system must be solved with the boundary conditions

$$P_1(t, 1-\epsilon\cos\omega_s t) = P_2(t, -1-\epsilon\cos\omega_s t) = 0 . \tag{13}$$

Equations (11) and (12) can be written as two decoupled uniform differential equations,

$$\frac{\partial P_{1,2}}{\partial t} = \frac{\partial}{\partial x}\left[D\frac{\partial P_{1,2}}{\partial x} + xP_{1,2} \right] = 0 . \tag{14}$$

At the points of switching between the states 1 and 2 solutions of these equations obey the boundary conditions

$$\left[\frac{\partial P_{1,2}}{\partial x} \right] = \mp \frac{\partial P_{2,1}}{\partial x} \quad \text{at } x = \mp 1 - \epsilon\cos\omega_s t , \tag{15}$$

where the square brackets denote the jump of a function,

$$[f(x)] \equiv f(x+0) - f(x-0) . \tag{16}$$

The populations $p_1(t)$ and $p_2(t)$ of the two states of the trigger are given (in the absence of a signal, $\epsilon=0$) by the integrals

$$p_1(t) = \int_{-\infty}^{1} P_1(x,t)dx , \tag{17}$$

$$p_2(t) = \int_{-1}^{\infty} P_2(x,t)dx . \tag{18}$$

The normalization condition looks like

$$p_1(t) + p_2(t) = 1 . \tag{19}$$

In the next section a steady-state solution of Eqs. (11) and (12) will be found. As a simplest time-dependent problem we consider then the process of relaxation of the populations $p_1(t)$ and $p_2(t)$ when at $t=0$ the trigger is switched into the state 1. In this way a fundamental quantity $w(t)$ is introduced which describes the distribution of the lifetime of the trigger states. In terms of the Fourier transform of $w(t)$ one can express the noise spectrum

$$N(\omega) = \int \langle p_1(t) - p_2(t), p_1(t') - p_2(t') \rangle \times \exp[i\omega(t-t')]dt . \tag{20}$$

In a similar manner, the small-signal application coefficient $A(\omega_s)$ will be calculated.

III. STEADY-STATE SOLUTION

To begin with, consider the trigger under effect of noise in the absence of a signal, $\epsilon=0$. In the steady state Eqs. (11) and (12) become ordinary differential equations,

$$\frac{d}{dx}\left[D\frac{dP_{1,2}}{dx} + xP_{1,2} \right] = -J_{2,1}\delta(x \pm 1) , \tag{21}$$

where

$$J_{1,2} = \mp D \left. \frac{dP_{1,2}}{dx} \right|_{x=\pm 1} \tag{22}$$

are (positive) fluxes of probabilities. Because of the symmetry of the problem, in the steady state

$$J_1 = J_2 \equiv J_0(D) , \tag{23}$$

and one needs only to solve one of the equations, which can be written as

$$D\frac{dP_1}{dx} + xP_1 = -J_0 \Theta(1-|x|) , \tag{24}$$

where $\Theta(z)$ is the Heavyside function. Its solution is

$$P_1(x) = \begin{cases} (J_0/D)e^{-x^2/2D} \int_x^1 e^{y^2/2D} dy , & -1 < x < 1 \quad (25) \\ (J_0/D)e^{-x^2/2D} \int_{-1}^1 e^{y^2/2D} dy , & x < -1 . \quad (26) \end{cases}$$

The flux J_0 is determined by the normalization condition,

$$\int_{-\infty}^{1} P_1(x)dx = 1/2 , \tag{27}$$

which gives

$$J_0(D)=\left[\frac{D}{8\pi}\right]^{1/2}\Big/\int_0^1 e^{y^2/2D}dy\ . \tag{28}$$

In the limiting cases,

$$J_0\approx(8\pi D)^{-1/2}\exp(-1/2D)\ ,\quad D\ll 1\ , \tag{29}$$

and

$$J_0\approx(D/8\pi)^{1/2}\ ,\quad D\gg 1\ . \tag{30}$$

The solution derived in this section will serve below as a zero-order approximation for considering more complicated problems. In order to render the presentation more transparent, it is useful to introduce some new concepts and notations. The points $x=1$ (for state 1) and $x=-1$ (for state 2) will be referred to as the sink points, whereas the points $x=-1$ (for state 1) and $x=1$ (for state 2) will be called the source points. The values of a function $f(x)$ at these points are denoted by subscripts

$$f_\pm\equiv f(\pm 1)\ , \tag{31}$$

whereas the jumps of a function will be denoted similarly to Eq. (16),

$$[f]_\pm\equiv f(\pm 1+0)-f(\pm 1-0)\ . \tag{32}$$

The solutions derived above will be denoted by $P_{1,2}^{(0)}(x)$. To calculate perturbatively the small-signal amplification coefficient, we need the derivatives of $P_{1,2}^{(0)}(x)$ at the sink points,

$$\left.\frac{dP_{1,2}^{(0)}}{dx}\right|_\pm=\mp\frac{J_0}{D}\ , \tag{33}$$

and the jumps of their derivatives at the source points,

$$\left[\frac{dP_{1,2}^{(0)}}{dx}\right]_\pm=\frac{J_0}{D}\ . \tag{34}$$

IV. RELAXATION OF POPULATION

Immediately after a switching of the trigger, the probability distribution $P_1(t,x)$ [or $P_2(t,x)$] is concentrated around $x=-1$ (or $x=1$),

$$P_1(0,x)=\delta(x+1)\ , \tag{35}$$

where the instant of the switching is taken as a reference point. For $t>0$, the distribution $P_1(t,x)$ spreads out and with increasing t approaches the equilibrium distribution [see Eqs. (25) and (26)]. In order to solve this time-dependent problem it is convenient to use the Laplace transformation

$$\int_0^\infty e^{-\lambda t}P_{1,2}(t,x)dt\equiv\exp[(1-x^2)/4D]\mathcal{P}_{1,2}(\lambda,x)\ . \tag{36}$$

The exponential factors cancel out in all the final expressions of this section. Therefore, they will be omitted in all the intermediate calculations. Introducing the fluxes,

$$J_{1,2}(\lambda)=\mp D\left.\frac{d\mathcal{P}_{1,2}(\lambda,x)}{dx}\right|_{x=\pm 1}\ , \tag{37}$$

the system (11) and (12) becomes

$$D\mathcal{P}_1''-(x^2/4D+\lambda-\tfrac{1}{2})\mathcal{P}_1=-[1+J_2(\lambda)]\delta(x+1)\ , \tag{38}$$

$$D\mathcal{P}_2''-(x^2/4D+\lambda-\tfrac{1}{2})\mathcal{P}_2=J_1(\lambda)\delta(x-1)\ . \tag{39}$$

The uniform equations for $\mathcal{P}_{1,2}$ have two linearly independent solutions, the parabolic cylinder functions, $U(\lambda-\frac{1}{2},x/D^{1/2})$ and $V(\lambda-\frac{1}{2},x/D^{1/2})$ [19]. In the final results the function $V(\lambda-\frac{1}{2},z)$ only enter through the Wronskian, so it is convenient to fix its amplitude by the condition

$$V\frac{dU}{dz}-U\frac{dV}{dz}=D^{1/2}\ . \tag{40}$$

For the sake of brevity, the following notations are introduced:

$$Y(x)\equiv U(\lambda-\tfrac{1}{2},x/D^{1/2})V_- - V(\lambda-\tfrac{1}{2},x/D^{1/2})U_-\ , \tag{41}$$

$$\psi(x)\equiv U(\lambda-\tfrac{1}{2},x/D^{1/2})\ . \tag{42}$$

In the sequel, we make use of the relations

$$Y_-=0\ , \tag{43}$$

$$Y'_-=1\ , \tag{44}$$

$$\psi Y'-Y\psi'=\psi_-\ . \tag{45}$$

In these notations the solution of Eqs. (38) and (39) continuous at the source points, $x=\pm 1$, is given by

$$\mathcal{P}_{1,2}(x)=\begin{cases}K_{1,2}Y(\mp x)\psi_+\ , & |x|<1 \quad (46)\\ K_{1,2}Y_+\psi(\mp x)\ , & |x|>1\ . \quad (47)\end{cases}$$

Calculating derivatives at the sink points, one obtains

$$\left.\frac{d\mathcal{P}_{1,2}}{dx}\right|_\pm=\mp K_{1,2}\psi_+\ , \tag{48}$$

whereas for the jumps of the derivatives at the source points holds

$$\left[\frac{d\mathcal{P}_{1,2}}{dx}\right]_\mp=-K_{1,2}\psi_-\ . \tag{49}$$

Substitution of these results into Eqs. (15) yields two equations for $K_{1,2}$,

$$K_1\psi_- - K_2\psi_+=1/D\ , \tag{50}$$

$$K_1\psi_+ - K_2\psi_-=0\ . \tag{51}$$

Solving this system and substituting the result into equation

$$J_{1,2}=DK_{1,2}\psi_+\ , \tag{52}$$

yields the expressions for the fluxes,

$$J_1=\frac{\psi_+\psi_-}{\psi_-^2-\psi_+^2}\ , \tag{53}$$

$$J_2=\frac{\psi_+^2}{\psi_-^2-\psi_+^2}\ . \tag{54}$$

In the limit $\lambda\to 0$, the fluxes $J_{1,2}$ approach the same value,

$$J_{1,2}\approx J_0/\lambda\ ,\quad \lambda\ll 1\ , \tag{55}$$

which gives the expression for J_0 in terms of the functions $\psi_\pm$,

$$J_0^{-1}=2\frac{d}{d\lambda}\frac{\psi_-(\lambda)}{\psi_+(\lambda)}\bigg|_{\lambda=0}\ . \tag{56}$$

The ratio ψ_-/ψ_+ is the most fundamental quantity in our theory. To demonstrate this statement, consider the decay of state 1 under condition that the flux $J_2=0$. Assuming $K_2=0$ in the system (50) and (51) gives the result

$$J_1(\lambda)=\frac{\psi_+(\lambda)}{\psi_-(\lambda)}\ . \tag{57}$$

The decay or the population $p_1(t)$ with time is described by the equation

$$\frac{dp_1(t)}{dt}=-w(t)\ , \tag{58}$$

and $w(t)=J_1(t)$ can be interpreted as the distribution of the lifetime of the trigger states: when sampling an ensemble of the triggers one finds that a trigger survives in a given state a time between t and $t+dt$ with the probability $w(t)dt$. The inverse Laplace transformation gives

$$w(t)=\int\frac{d\omega}{2\pi}e^{-i\omega t}\frac{U(-i\omega-\frac{1}{2},D^{-1/2})}{U(-i\omega-\frac{1}{2},-D^{-1/2})}\ ,\quad t>0\ , \tag{59}$$

where $U(a,z)$ is the parabolic cylinder function. In what follows, the notation that will be used is

$$w(\lambda)\equiv\int_0^\infty e^{-\lambda t}w(t)dt=\frac{\psi_+}{\psi_-}=\frac{U(\lambda-\frac{1}{2},D^{-1/2})}{U(\lambda-\frac{1}{2},-D^{-1/2})}\ . \tag{60}$$

V. SPECTRUM OF THE OUTPUT NOISE

The results obtained above enable one to calculate the spectrum of the noise (20). In order to achieve this goal, we resort to Eq. (58). Physically, this equation describes the stochastic behavior of the ST as a series of switching events between states 1 and 2 with the time t between the switches distributed with the probability density $w(t)$. Equation (20) is equivalent to the relation

$$N(\omega)=\frac{1}{\omega^2}\int\langle j(t)j(t')\rangle\exp[i\omega(t-t')]dt\ , \tag{61}$$

where

$$j(t)\equiv d(p_1-p_2)/dt\ . \tag{62}$$

At a switching event the value of $p_1(t)-p_2(t)$ changes by ± 2, so that $j(t)$ is a sum of δ-like pulses with the amplitudes ± 2,

$$j(t)=2\sum_{n=1}^{\infty}(-1)^n\delta\left[t-\sum_{l=1}^{n}t_l\right]\ . \tag{63}$$

The lth switching event occurs in the interval of time (t_l,t_l+dt_l) after the $(l-1)$th event with the probability $w(t_l)dt_l$. The intervals t_l $(l=1,\dots,\infty)$ are completely uncorrelated between each other, since the trigger completely forgets its history after each switching event. Therefore, to average a function of $j(t)$, one needs to integrate it over all the t_l's with the weights $w(t_l)$. Substitution of Eq. (63) into Eq. (61) and integration over t_1 with the weight $w(t_1)$ yields

$$N(\omega)=\frac{4}{\omega^2}\sum_{n=2}^{\infty}\left\langle w\left[t-\sum_{l=2}^{n}t_l\right]\left\{1+\sum_{m>n}^{\infty}(-1)^{n-m}\left[\exp\left[-i\omega\sum_{l=n+1}^{m}t_l\right]+\text{c.c.}\right]\right\}\right\rangle\ , \tag{64}$$

where c.c. denote the complex conjugation. To get rid of transient effects, one should assume that $t\to\infty$, whereas ω is finite. In this case to the sum (64) contribute finite values of $m-n$, whereas the sum over n becomes singular when $t\to\infty$. Hence, the two factors in the angular brackets in Eq. (64) can be averaged independently. Averaging of the exponent gives

$$W_k(\omega)=\left\langle\exp\left[-i\omega\sum_{l=n+1}^{n+k}t_l\right]\right\rangle=\left[\int_0^\infty e^{-i\omega t}w(t)dt\right]^k=w^k(-i\omega)\ , \tag{65}$$

$$W_{-k}=w^k(i\omega)\ . \tag{66}$$

Summation over k then yields

$$\sum_{k=-\infty}^{\infty}(-1)^kW_k(\omega)=1+\sum_{k=1}^{\infty}(-1)^k[w^k(i\omega)+w^k(-i\omega)]=\text{Re}\frac{1-w(i\omega)}{1+w(i\omega)}\ . \tag{67}$$

The sum over n after averaging of the first factor in Eq. (64) looks like

$$C=\sum_{n=2}^{\infty}\left\langle w\left[t-\sum_{l=2}^{n}t_l\right]\right\rangle=\int\frac{d\Omega}{2\pi}e^{i\Omega t}\sum_{n=1}^{\infty}w^n(i\Omega)=\int\frac{d\Omega}{2\pi}\frac{\exp(i\Omega t)w(i\Omega)}{1-w(i\Omega)}\ . \tag{68}$$

For $t\to\infty$, the main contribution into the integral comes from $|\Omega|\ll 1$, so that one can use the expansions

$$w(i\Omega)\approx 1 \tag{69}$$

and

$$1-w(i\Omega)\approx i\Omega/2J_0 , \tag{70}$$

as it follows from Eqs. (56) and (60). The final result for C is

$$C=2J_0 \lim_{t\to\infty} \int \frac{d\Omega}{2\pi}\frac{\exp(i\Omega t)}{i\Omega+0}=2J_0 . \tag{71}$$

Now, the expression for the noise spectrum takes on a very simple form,

$$N(\omega)=\frac{8J_0}{\omega^2}\mathrm{Re}\frac{\psi_- - \psi_+}{\psi_- + \psi_+}=\frac{8J_0}{\omega^2}\mathrm{Re}\frac{U(i\omega-\frac{1}{2},-D^{-1/2})-U(i\omega-\frac{1}{2},D^{-1/2})}{U(i\omega-\frac{1}{2},-D^{-1/2})+U(i\omega-\frac{1}{2},D^{-1/2})} , \tag{72}$$

where J_0 is given by Eq. (28) and $U(a,z)$ is the parabolic cylinder function. With the use of asymptotic functions

$$U(-i\omega-\tfrac{1}{2},z)\approx\exp(-z^2/4) , \quad z\gg 1 , \tag{73}$$

and

$$U(-i\omega-\tfrac{1}{2},z)\approx\exp(-z^2/4)-i\omega(2\pi)^{1/2}z^{-1}\exp(z^2/4) , \quad -z\gg 1 , \tag{74}$$

which are valid for $|\omega|\ll 1$, one obtains

$$N(\omega)\approx\frac{2\gamma}{\omega^2+\gamma^2} , \quad D\ll 1 , \tag{75}$$

where

$$\gamma\approx 4J_0\approx\left[\frac{2}{\pi D}\right]^{1/2}\exp\left[-\frac{1}{2D}\right] . \tag{76}$$

In the opposite limit of a strong noise,

$$N(\omega)=\frac{2}{\pi\omega^2}\mathrm{Re}\,i\omega B(\tfrac{1}{2},\tfrac{1}{2}+i\omega/2) , \quad D\gg 1 , \tag{77}$$

where $B(x,y)$ is the Euler beta function. Asymptotically,

$$N(\omega)\approx\begin{cases} 2\ln 2 , \quad \omega\ll 1 , \ D\gg 1 & (78)\\ \dfrac{2}{\pi^{1/2}}\dfrac{1}{|\omega|^{3/2}} , \quad D\gg|\omega|\gg 1 . & (79)\end{cases}$$

With the use of the inverse Fourier transformation Eq. (77) yields

$$N(t)=\frac{2}{\pi}\arcsin[\exp(-|t|)] , \quad D\gg 1 . \tag{80}$$

Note that $N(t)$ is expandable in a sum of $\exp[-(2n+1)|t|]$ with integer n's, in accord with the fact that the eigenvalues λ of the Schrödinger equation with parabolic potential are integers.

VI. SMALL-SIGNAL AMPLIFICATION COEFFICIENT

Under the effect of a sinusoidal input, the output of the ST changes in two ways. On the one hand, it acquires a sinusoidal component with the input frequency, which is called the signal. On the other hand, the broad output spectrum is also changed. We will be interested in calculation of a small-signal amplification coefficient. A δ-like contribution to the output spectrum has its origin in coherency of the state populations $p_1(t)-p_2(t)$ and $p_1(t')-p_2(t')$ at $|t-t'|\to\infty$. In this limit, the averaging of the two factors in Eq. (16) can be performed independently with the result

$$S(\omega)=\int\langle p_1(t)-p_2(t)\rangle\langle p_1(t')-p_2(t')\rangle \times\exp[i\omega(t-t')]dt =\frac{2\pi}{\omega_s^2}|j(\omega_s)|^2\delta(\omega-\omega_s) , \tag{81}$$

where $j(\omega_s)$ is the amplitude of the flux induced in the trigger by a small signal $\epsilon\cos\omega_s t$,

$$\langle p_1(t)-p_2(t)\rangle=\frac{ij(\omega_s)}{2\omega_s}\exp(-i\omega_s t)+\mathrm{c.c.} \tag{82}$$

Hence, to calculate the amplification coefficient it is sufficient to find the relation between the output $j(\omega_s)$ and the amplitude ϵ of a small signal $\epsilon\exp(-i\omega_s t)$.

To begin with, this problem is considered in a phenomenological approach, which is justified in the limit of a weak noise, $D\ll 1$. In this case transitions between states 1 and 2 are exponentially rare, $\tau_0\propto\exp[1/(2D)]\gg 1$ (τ_0 is the lifetime of a Brownian particle in a parabolic potential well with the barrier height $\frac{1}{2}$ and the noise intensity $D\ll 1$). If the signal frequency is small compared to the friction coefficient, $\omega_s\ll 1$, the switching of the Schmitt trigger can be described in an adiabatic approximation. This approach implicates that the lifetimes of the two states of the trigger, $\tau_1(t)$ and $\tau_2(t)$, are calculated for the barrier heights frozen at the moment t, and then they are used as the rate coefficients in equations for populations $p_1(t)$ and $p_2(t)$. A small signal $\epsilon\exp(-i\omega_s t)$, $\epsilon\ll 1$, changes the barrier heights only slightly,

$$\frac{1}{\tau_{1,2}(t)}=\frac{1}{2\tau_0}\exp\left\{-\frac{[1\mp\epsilon\exp(-i\omega_s t)]^2-1}{2D}\right\} \approx\frac{\exp[\pm\epsilon\exp(-i\omega_s t)/D]}{2\tau_0} , \tag{83}$$

where the coefficient $\frac{1}{2}$ takes into account that, in the case of the Schmitt trigger, the Brownian particle escapes only across one barrier. The equation for the state populations, $p_1(t)$ and $p_2(t)$,

$$\frac{dp_{1,2}}{dt}=-\frac{p_{1,2}}{\tau_{1,2}(t)}+\frac{p_{2,1}}{\tau_{2,1}(t)} , \tag{84}$$

can be simplified, if the signal is considered in a linear approximation, which is justified for

$$\epsilon \ll D . \tag{85}$$

This inequality yields the criterion of a small-signal magnitude in the limit of a weak noise. Expanding $\tau_{1,2}(t)$ linearly in ϵ and substituting

$$p_{1,2}=\tfrac{1}{2}\pm p , \tag{86}$$

one obtains the equation

$$\frac{dp}{dt}=-\frac{p}{\tau_0}-\frac{\epsilon\exp(-i\omega_s t)}{2D\tau_0} . \tag{87}$$

The solution of this equation is straightforward, and the result for $j(\omega_s)$ looks like

$$j(\omega_s)=\frac{\epsilon i\omega_s}{D(1-i\omega_s\tau_0)} , \tag{88}$$

since $dp_1/dt-dp_2/dt=2dp/dt$. Introducing the amplification coefficient, $A(\omega_s)$, by the relation

$$S(\omega)=\epsilon^2 A(\omega_s)\delta(\omega-\omega_s) , \tag{89}$$

one obtains the following expression:

$$A(\omega_s)=\frac{\pi}{2D^2}\frac{1}{1+\omega_s^2\tau_0^2} . \tag{90}$$

In order to investigate the general case, $D\sim 1$, one has to solve the system of two diffusion equations. As before, a linear response to a small signal, $\epsilon(t)=\epsilon\exp(-i\omega_s t)$, is considered. The result for the cosine signal is then obtained by restoring simple numeric factors. Expanding $P_{1,2}(t,x)$ in $\epsilon(t)$,

$$P_{1,2}(t,x)=P^0_{1,2}(x)+\epsilon(t)\exp[(1-x^2)/4D]r_{1,2}(x) , \tag{91}$$

one obtains for the functions $r_{1,2}$ the solution very similar to Eqs. (46) and (47) up to the substitution of λ by $-i\omega_s$. In contrast to Sec. IV, one has to change the boundary conditions for this solution. Since $P^{(0)}_{1,2}(\pm 1)=0$, from Eq. (13) it follows that

$$P_{1,2}[t,\pm 1-\epsilon(t)]\approx-\epsilon(t)\left.\frac{dP^{(0)}_{1,2}}{dx}\right|_{\pm}+\epsilon(t)r_{1,2}|_{\pm}\approx 0 , \tag{92}$$

which is equivalent to the condition [see Eq. (33)]

$$r_{1,2}|_{\pm}=\mp J_0/D . \tag{93}$$

With an account of this condition we write down

$$r_{1,2}(x)=k_{1,2}Y(\mp x)\psi_+\mp\frac{J_0}{D}\frac{\psi(\mp x)}{\psi_-} , \quad |x|<1 . \tag{94}$$

To write down $r_{1,2}(x)$ for $|x|>1$, one has to use the conditions of continuity of $P_{1,2}(t,x)$ at the source points, $x=\mp 1-\epsilon(t)$. Expanding $P^{(0)}_{1,2}(x)$ linearly near these points, one obtains

$$[r_{1,2}]_{\mp}=\left[\frac{dP^{(0)}_{1,2}}{dx}\right]_{\mp}=\frac{J_0}{D} . \tag{95}$$

This condition is sufficient to construct the expression

$$r_{1,2}(x)=\left[k_{1,2}Y_+\mp\frac{J_0}{D}\frac{\psi_+-\psi_-}{\psi_+\psi_-}\right]\psi(\mp x) , \quad |x|>1 . \tag{96}$$

Below, the derivatives of $r_{1,2}(x)$ at the sink points,

$$\left.\frac{dr_{1,2}}{dx}\right|_{\pm}=\mp k_{1,2}\psi_+\frac{J_0}{D}\frac{\psi'_-}{\psi_-} , \tag{97}$$

and the jumps of these derivatives at the source points,

$$\left[\frac{dr_{1,2}}{dx}\right]_{\mp}=-k_{1,2}\psi_-\pm\frac{J_0}{D}\frac{\psi'_+}{\psi_+} , \tag{98}$$

will be used to calculate the coefficients $k_{1,2}$. To this end, one needs to derive expressions for the quantities entering Eqs. (15). In an approximation linear in ϵ,

$$\begin{aligned}\left.\frac{dP_{1,2}}{dx}\right|_{\pm 1-\epsilon(t)}&\approx\frac{d}{dx}\left.\left[P^{(0)}_{1,2}(x)+\epsilon(t)\exp\left[\frac{1-x^2}{4D}\right]r_{1,2}(x)\right]\right|_{\pm 1-\epsilon(t)}\\&\approx\left.\frac{dP^{(0)}_{1,2}}{dx}\right|_{\pm}-\epsilon(t)\left.\frac{d^2P_{1,2}}{dx^2}\right|_{\pm}\mp\frac{\epsilon(t)}{2D}r_{1,2}|_{\pm}+\epsilon(t)\left.\frac{dr_{1,2}}{dx}\right|_{\pm} .\end{aligned} \tag{99}$$

With the use of the relation

$$\left.\frac{d^2P^{(0)}_{1,2}}{dx^2}\right|_{\pm}=\mp\frac{1}{D}\left.\frac{dP^{(0)}_{1,2}}{dx}\right|_{\pm}=\frac{J_0}{D^2} , \tag{100}$$

which follows from the static diffusion equation for the points of vanishing $P_{1,2}(x)$, and Eqs. (93) and (97), one obtains a simple expression,

$$\left.\frac{dP_{1,2}}{dx}\right|_{\pm 1-\epsilon(t)}\approx\mp\frac{J_0}{D}+\epsilon(t)\left[\mp k_{1,2}\psi_+ +\frac{J_0}{D}\left[\frac{\psi'_-}{\psi_-}-\frac{1}{2D}\right]\right] . \tag{101}$$

The jumps of the derivatives are calculated in the same way. Similarly to the second line of Eq. (99), one arrives at the expression

$$\left[\frac{dP_{1,2}}{dx}\right]_{\mp 1-\epsilon(t)} \approx \left[\frac{dP_{1,2}^{(0)}}{dx}\right]_{\mp} - \epsilon(t)\left[\frac{d^2P_{1,2}}{dx^2}\right]_{\mp} \pm \frac{\epsilon(t)}{2D}[r_{1,2}]_{\mp} + \epsilon(t)\left[\frac{dr_{1,2}}{dx}\right]_{\mp} . \quad (102)$$

Making use of the relation

$$\left[\frac{d^2P_{1,2}^{(0)}}{dx^2}\right]_{\mp} = \mp\frac{J_0}{D^2} , \quad (103)$$

which is obtained analogously to Eq. (100), and substituting Eqs. (95) and (98), yields the result

$$\left[\frac{dP_{1,2}}{dx}\right]_{\mp 1-\epsilon(x)} \approx -\frac{J_0}{D} + \epsilon(t)\left[-k_{1,2}\psi_- \pm \frac{J_0}{D}\left[\frac{\psi'_+}{\psi_+} + \frac{1}{2D}\right]\right] . \quad (104)$$

Substituting these expressions into Eqs. (15) and separating the terms linear in ϵ, one derives the system of equations

$$k_1\psi_- - k_2\psi_+ = \frac{J_0}{D}\left[\frac{\psi'_+}{\psi_+} + \frac{\psi'_-}{\psi_-}\right] , \quad (105)$$

$$k_1\psi_+ - k_2\psi_- = \frac{J_0}{D}\left[\frac{\psi'_+}{\psi_+} + \frac{\psi'_-}{\psi_-}\right] , \quad (106)$$

with the solutions

$$k_1 = -k_2 = \frac{J_0}{D}\frac{1}{\psi_+ + \psi_-}\left[\frac{\psi'_+}{\psi_+} + \frac{\psi'_-}{\psi_-}\right] . \quad (107)$$

The contributions to dp_1/dt linear in ϵ come in from both the sink point and the source point and are equal in the magnitude and sign. On the other hand, their sum is equal to $-dp_2/dt$. Hence, in order to find $j(\omega_s)$, one only needs to substitute k_1 into Eq. (101) and then multiply it by $4D$,

$$j(\omega_s) = 4\epsilon J_0\left[\frac{\psi'_-}{\psi_+ + \psi_-} - \frac{1}{2D}\right] . \quad (108)$$

The small-signal amplification coefficient is then given by

$$A(\omega_s) = \frac{8\pi J_0^2}{\omega_s^2}\left|\frac{\psi_- - \psi_+}{\psi_- + \psi_+} - \frac{1}{2D}\right|^2 . \quad (109)$$

In the limit $D \ll 1$, one has the asymptotics

$$\psi_+ \approx \exp(-1/4D) , \quad (110)$$

$$\psi_- \approx (1 - 2i\omega_s\tau_0)\psi_+ , \quad (111)$$

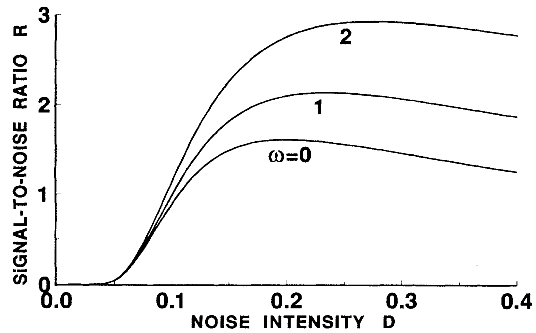

FIG. 2. The signal-to-noise ratio $R(\omega,D)$ plotted against the noise intensity D for several different signal frequencies ω.

$$\psi'_+ \approx -\psi_+/2D , \quad (112)$$

$$\psi'_- \approx (1 + 2i\omega_s\tau_0)\psi_+/2D , \quad (113)$$

where

$$\tau_0 \approx \frac{1}{4J_0} \approx \left[\frac{\pi D}{2}\right]^{1/2} \exp\left[\frac{1}{4D}\right] . \quad (114)$$

Substitution of this expressions into Eq. (108) reproduces the result (88) of the phenomenological approach. In the opposite limit of a strong noise the amplification coefficient is independent of the frequency,

$$A(\omega_s) \approx 1/D , \quad D \gg 1 . \quad (115)$$

The signal-to-noise ratio,

$$R(\omega,D) \equiv A(\omega)/N(\omega) , \quad (116)$$

is plotted in Fig. 2 as a function of D for several values of ω. One sees that a pronounced maximum occurs at $D \sim 0.2 - 0.3$, which is a clear manifestation of SR.

VII. QUALITATIVE CLASSIFICATION OF THE NOISE MODELS

In the previous sections, we have considered in detail a particular model of SR, when a Schmitt trigger is pumped by a colored Gaussian noise, governed by Eq. (8). This equation describes motion of an overdamped Brownian particle in a parabolic potential. The presented approach can easily be extended to investigation of a whole class of stochastic resonance models, if one assumes that the noise fed into the Schmitt trigger is simulated by the Brownian motion in a potential $V(x)$,

$$\frac{dx}{dt} = -\frac{dV(x)}{dx} + (2D)^{1/2}\xi(t) . \quad (117)$$

For the sake of simplicity, the potential is assumed to be symmetric, $V(x) = V(-x)$. The diffusion equation in this case looks like

$$\frac{\partial P}{\partial t} = \frac{\partial}{\partial x}\left[D\frac{\partial P}{\partial x} + \frac{dV(x)}{dx}P\right] . \quad (118)$$

Without loss of generality, one assumes that the switching event occurs immediately as soon as the Brownian particle reaches the point $x = 1$ (for state 1) or the point

$x=-1$ (for state 2). Introducing the solution of the Schrödinger equation,

$$D\frac{d^2\psi}{dx^2}-\left[\frac{1}{4D}\left[\frac{dV}{dx}\right]-\frac{1}{2}\frac{d^2V}{dx^2}\right]\psi=i\omega\psi\ , \tag{119}$$

with the boundary condition

$$\psi(\omega,x)\to 0\ ,\quad x\to\infty\ , \tag{120}$$

the results of the previous sections may be rewritten in the form

$$N(\omega)=\frac{8J_0}{\omega^2}\mathrm{Re}\frac{\psi_- - \psi_+}{\psi_- + \psi_+} \tag{121}$$

and

$$A(\omega_s)=\frac{8\pi J_0^2}{\omega_s^2}\left|\frac{\psi'_- - \psi'_+}{\psi_- + \psi_+}+\frac{V'_-}{2D}\right|^2 , \tag{122}$$

where $J_0=J_0(D)$ is the steady-state flux of probability,

$$J_0(D)=\frac{D}{2}\left[\int_0^\infty e^{-V(x)/D}dx\int_0^1 e^{V(y)/D}dy\right]^{-1} , \tag{123}$$

and our standard notations are used,

$$\psi_\pm\equiv\psi(\omega,\pm 1)\ , \tag{124}$$

$$\psi'_\pm\equiv\left.\frac{d\psi(\omega,x)}{dx}\right|_{\pm 1} . \tag{125}$$

In the limit of a weak noise, $D\ll 1$, all the models of noise give the identical results for $N(\omega)$ and $A(\omega)$ [see Eqs. (72) and (90)], if the decay rate is determined by the relation

$$\gamma\equiv\frac{1}{\tau_0}=V'_+\left[\frac{2V''_+}{\pi D}\right]^{1/2}\exp\left[-\frac{V_+}{D}\right] . \tag{126}$$

These results show that, for small D, the signal-to-noise ratio grows exponentially with D. This behavior is universal, since it does not depend on the particular shape of the potential as long as $V_+\gg D$.

The range of intermediate D, when $V_+\sim D$, can only be solved numerically. Simplification of calculations is again possible in the case of a strong noise, when scaling arguments can be successfully used for a very important class of the potentials $V(x)$ with powerlike asymptotics,

$$V(x)\approx V_{\mathrm{inf}}x^\alpha\ ,\quad x\to\infty\ . \tag{127}$$

Then, in a limit of strong noise from Eq. (123) follows

$$J_0\propto D^{1-1/\alpha}\ ,\quad D\gg 1\ . \tag{128}$$

Solution of Eq. (119) for $x\gg 1$ can be written in the form

$$\psi(\omega,x)=\tilde\psi(\omega/D^{1-2/\alpha},x/D^{1/\alpha})\ , \tag{129}$$

where the function $\tilde\psi$ obeys an equation which does not contain D. For $\alpha\geq 2$, solutions of this equation can be continued down to the points of switching, since in the region $|V(x)|<D$ the potential can be neglected, so that its particular shape in this region becomes irrelevant. The switching points, $x=\pm 1$, are placed very closely on the scale of $x\sim D^{1/\alpha}\gg 1$. Therefore, in Eq. (122) one has

$$\psi'_- - \psi'_+\propto D^{-2/\alpha}\psi_\pm\geq\psi_\pm/D\ . \tag{130}$$

With an account of these dependences, for the noise and signal spectra one obtains

$$N(\omega)=D^{2/\alpha-1}n_\alpha\left[\frac{\omega}{D^{1-2/\alpha}}\right] , \tag{131}$$

$$A(\omega)=D^{-2/\alpha}a_\alpha\left[\frac{\omega}{D^{1-2/\alpha}}\right] , \tag{132}$$

where $n_\alpha(z)$ and $a_\alpha(z)$ are functions, dependent on α as a parameter. The signal-to-noise ratio is then given by

$$\frac{A(\omega)}{N(\omega)}=D^{1-4/\alpha}f_\alpha\left[\frac{\omega}{D^{1-2/\alpha}}\right] , \tag{133}$$

where the function $f_\alpha(z)$ is determined by the parameter α. The last equation shows that $\alpha=4$ is a separating point in the sense that for $\alpha>4$ the signal-to-noise ratio increases with D both for small and large values of D, and there are no grounds to expect a maximum at a $D\sim 1$. On the contrary, for $\alpha<4$ the signal-to-noise ratio rises exponentially with D for $D\ll 1$ and fall down with D for $D\gg 1$. Therefore, the signal-to-noise ratio must have a maximum at a $D\sim 1$. The power $\alpha=4$ may be considered as a rather large, so that the phenomenon of SR is observable for a wide class of the noise models. Qualitatively, for flat potentials (small α) with increasing D the distribution of particles around the switching points becomes depleted which tends to suppress the switchings. Steeper potentials (large α) confine particles to a narrow region near $x\sim 1$ enhancing amplification of the signal.

In conclusion, two different approaches were used above to investigate SR in the Schmitt trigger. The major part of this article considers the "case" when the Schmitt trigger is fed by a noise represented by the Ornstein-Uhlenbeck process. Our exact analytical solution is tightly restricted to this assumption. The main result here is that the Schmitt trigger, indeed, exhibit SR for this type of noise. In the rest of the paper, a qualitative consideration is given for a class of more general models of noise. It is shown that the signal-to-noise ratio increases monotonically with the noise strength if the noise amplitude distribution remains sufficiently narrow.

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[1] (a) R. Benzi, A. Sutera, and A. Vulpiani, J. Phys. A **14**, L453 (1981); (b) R. Benzi, G. Parisi, A. Sutera, and A. Vulpiani, Tellus **34**, 10 (1982); (c) SIAM J. Appl. Math. **43**, 565 (1983).

[2] (a) C. Nicolis, Tellus **34**, 1 (1982); (b) in *Proceedings of the NATO ARW on Stochastic Resonance in Physics and Biology*, edited by F. Moss, A. Bulsara, and M. F. Shlesinger [J. Stat. Phys. **70**, 1 (1993)].

[3] S. Fauve and F. Heslot, Phys. Lett. **97A**, 5 (1983).

[4] (a) B. McNamara, K. Wiesenfeld, and R. Roy, Phys. Rev. Lett. **60**, 2626 (1988); (b) G. Vermuri and R. Roy, Phys. Rev. A **39**, 4668 (1989).

[5] B. McNamara and K. Wiesenfeld, Phys. Rev. A **39**, 4854 (1989).

[6] (a) P. Jung and P. Hänggi, Europhys. Lett. **8**, 505 (1989); (b) Phys. Rev. A **41**, 2977 (1990); (c) **44**, 8032 (1991).

[7] P. Jung, Z. Phys. B **16**, 521 (1989).

[8] T. Zhou, F. Moss, and P. Jung, Phys. Rev. A **42**, 3161 (1990).

[9] L. Gammaitoni, F. Marchesoni, E. Menichella-Saetta, and S. Santucci, Phys. Rev. Lett. **62**, 349 (1989).

[10] H. Gang, G. Nicolis, and C. Nicolis, Phys. Rev. A **42**, 2030 (1990).

[11] I. Dayan, M. Gitterman, and G. H. Weiss, Phys. Rev. A **46**, 757 (1992).

[12] M. Dykman, P. V. E. McClintock, R. Mannella, and N. Stocks, Pis'ma Zh. Eksp. Teor. Fiz. **52**, 783 (1990) [JETP Lett. **52**, 145 (1990)].

[13] L. Gammaitoni, M. Martinelli, L. Pardi, and S. Santucci, Phys. Rev. Lett. **67**, 1799 (1991).

[14] T. Zhou and F. Moss, Phys. Rev. A **41**, 4255 (1990).

[15] (a) H. Gang, G. Qing, D. Gong, and X. Weng, Phys. Rev. A **44**, 6414 (1991); (b) G. De-chun, H. Gang, W. Xiao-dong, Y. Chun-yan, Q. Guang-rong, L. Rong, and D. Da-fu, *ibid.* **46**, 3243 (1992); (c) H. Gang, G. De-chun, W. Xiao-dong, Y. Chun-yuan, Q. Guang-rong, and L. Rong, *ibid.* **46**, 3250 (1992).

[16] *Proceedings of the NATO ARW on Stochastic Resonance in Physics and Biology* [Ref. 2(b)].

[17] *Proceedings of the NATO ARW, Rate Processes in Dissipative Systems: 50 Years after Kramers,* edited by P. Hänggi and J. Troe [Ber. Bunsenges. Phys. Chem. **95**, (1991)].

[18] F. Moss, in *Some Problems in Statistical Physics,* edited by George Weiss, Frontiers in Applied Mathematics (SIAM, Philadelphia, in press).

[19] *Handbook of Mathematical Functions,* edited by M. Abramowitz and Irene A. Stegun (Dover, New York, 1970).